THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4060 Complex Analysis 2022-23
Tutorial 8
23th March 2022

1. (Ex.7 Ch.8 in textbook) Provide all the details in the proof of the formula for the solution of the Dirichlet problem in a strip discussed in Section 1.3. Recall that it suffices to compute the solution at the points $z = iy$ with $0 < y < 1$.

   (a) Show that if $e^{i\theta} = G(iy)$, then

   $$re^{i\theta} = i\frac{\cos \pi y}{1 + \sin \pi y}$$

   This leads to two separate cases: either $0 < y \leq \frac{1}{2}$ and $\theta = \frac{\pi}{2}$, or $\frac{1}{2} < y \leq 1$ and $\theta = -\frac{\pi}{2}$. In either case, show that

   $$r^2 = \frac{1 - \sin \pi y}{1 + \sin \pi y} \quad \text{and} \quad P_r(\theta - \phi) = \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}$$

   (b) In the integral $\frac{1}{2\pi}\int_0^{\pi} P_r(\theta - \varphi)\tilde{f}_0(\varphi)d\varphi$ make the change of variables $t = F(e^{i\varphi})$. Observe that

   $$e^{i\varphi} = \frac{i - e^{\pi t}}{i + e^{\pi t}},$$

   and then take the imaginary part and differentiate both sides to establish the two identities:

   $$\sin \varphi = \frac{1}{\cosh \pi t} \quad \text{and} \quad \frac{d\varphi}{dt} = \frac{\pi}{\cosh \pi t}.$$

   Hence deduce that

   $$\begin{aligned}\frac{1}{2\pi}\int_0^{\pi} P_r(\theta - \varphi)\tilde{f}_0(\varphi)d\varphi &= \frac{1}{2\pi}\int_0^{\pi} \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}\tilde{f}_0(\varphi)d\varphi \\ &= \frac{\sin \pi y}{2}\int_{-\infty}^{\infty} \frac{f_0(t)}{\cosh \pi t - \cos \pi y}dt\end{aligned}$$

   (c) Use a similar argument to prove the formula for the integral $\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta - \varphi)\tilde{f}_1(\varphi)d\varphi$.

Solution. (a) Recall the function $F : \mathbb{D} \to \Omega$ and $G : \Omega \to \mathbb{D}$ are defined as following:

$$F(w) = \frac{1}{\pi}\log(i\frac{1 - w}{1 + w}) \quad \textit{and} \quad G(z) = \frac{i - e^{\pi z}}{i + e^{\pi z}}$$

Thus if $re^{i\theta} = G(iy)$, then

$$\begin{aligned} re^{i\theta} &= \frac{i - e^{\pi iy}}{i + e^{\pi iy}} \\ &= \frac{i - \cos \pi y - i \sin \pi y}{i + \cos \pi y + i \sin \pi y} \\ &= \frac{-\cos \pi y + i(1 - \sin \pi y)}{\cos \pi y + i(1 + \sin \pi y)} \\ &= \frac{(-\cos \pi y + i(1 - \sin \pi y))(\cos \pi y - i(1 + \sin \pi y))}{\cos^2 \pi y + (1 + \sin \pi y)^2} \\ &= \frac{2i \cos \pi y}{\cos^2 \pi y + (1 + \sin \pi y)^2} \\ &= \frac{i \cos \pi y}{1 + \sin \pi y} \end{aligned}$$

Then we have:

$$\begin{aligned} r^2 &= \frac{(\cos \pi y)^2}{(1 + \sin \pi y)^2} \\ &= \frac{1 - \sin \pi y}{1 + \sin \pi y} \quad \text{and} \\ r \sin \theta &= \frac{\cos \pi y}{1 + \sin \pi y} \quad \text{and} \\ r \cos \theta &= 0 \\ P_r(\theta - \varphi) &= \frac{1 - r^2}{1 - 2r \cos(\theta - \varphi) + r^2} \quad \text{and} \\ &= \frac{1 - r^2}{1 - 2r(\cos \theta \cos \varphi + \sin \theta \sin \varphi) + r^2} \\ &= \frac{1 - r^2}{1 - 2r \sin \theta \sin \varphi + r^2} \\ &= \frac{1 - \frac{1 - \sin \pi y}{1 + \sin \pi y}}{1 - 2\frac{\cos \pi y}{1 + \sin \pi y} \sin \varphi + \frac{1 - \sin \pi y}{1 + \sin \pi y}} \\ &= \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}. \end{aligned}$$

(b) By direct computation,

$$\begin{aligned} e^{i\varphi} &= \frac{i - e^{\pi t}}{i + e^{\pi t}} \\ &= -\frac{-1 - 2ie^{\pi t} + e^{2\pi t}}{1 + e^{2\pi t}} \\ &= -\frac{e^{2\pi t} - 1}{1 + e^{2\pi t}} + i\frac{2e^{\pi t}}{1 + e^{2\pi t}} \\ &= -\tanh \pi t + i\frac{1}{\cosh \pi t} \end{aligned}$$

by comparing the real and imaginary part on both sides, we get

$$\cos\varphi = -\tanh \pi t \quad \text{and} \quad \sin\varphi = \frac{1}{\cosh \pi t}$$

differentiating $\sin\varphi$,

$$\cos\varphi \cdot \frac{d\varphi}{dt} = \frac{-\pi \sinh \pi t}{(\cosh \pi t)^2}$$

Thus

$$\frac{d\varphi}{dt} = \frac{\pi}{\cosh \pi t}$$

Then the fomula becomes

$$\begin{aligned}\frac{1}{2\pi}\int_0^{\pi} P_r(\theta-\varphi)\tilde{f}_0(\varphi)d\varphi &= \frac{1}{2\pi}\int_0^{\pi} \frac{\sin \pi y}{1-\cos \pi y \sin\varphi}\tilde{f}_0(\varphi)d\varphi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin \pi y}{1-\cos \pi y \frac{1}{\cosh \pi t}} f_0(t)\frac{d\varphi}{dt}dt \\ &= \frac{\sin \pi y}{2}\int_{-\infty}^{\infty} \frac{1}{\cosh \pi t - \cos \pi y} f_0(t)dt\end{aligned}$$

(c) In this case, we have

$$e^{i\varphi} = \frac{i-e^{\pi(t+i)}}{i+e^{\pi(t+i)}} = \frac{i+e^{\pi t}}{i-e^{\pi t}}$$

by similar argument, we have

$$\cos\varphi = -\tanh \pi t \quad \text{and} \quad \sin\varphi = -\frac{1}{\cosh \pi t}$$

and

$$\frac{d\varphi}{dt} = -\frac{\pi}{\cosh \pi t}$$

Thus

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta-\varphi)\tilde{f}_1(\varphi)d\varphi &= \frac{1}{2\pi}\int_{-\pi}^{0} \frac{\sin \pi y}{1-\cos \pi y \sin\varphi}\tilde{f}_1(\varphi)d\varphi \\ &= \frac{1}{2\pi}\int_{\infty}^{-\infty} \frac{\sin \pi y}{1-\cos \pi y \frac{-1}{\cosh \pi t}} f_1(t)\frac{d\varphi}{dt}dt \\ &= \frac{\sin \pi y}{2}\int_{-\infty}^{\infty} \frac{1}{\cosh \pi t + \cos \pi y} f_1(t)dt\end{aligned}$$

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2. (Ex.7 Ch.11 in textbook)Show that if $f : D(0,R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \le M$ for some $M > 0$, then

$$\left|\frac{f(z)-f(0)}{M^2-\overline{f(0)}f(z)}\right| \le \frac{|z|}{MR}.$$

Solution. If we want to use Schwarz lemma, we need to construct a function from $\mathbb{D}$ to $\mathbb{D}$ sending 0 to 0. Consider

$$\varphi_1(z_1) = R \cdot z_1 : \mathbb{D} \to D(0, R)$$

$$\varphi_2(z_2) = \frac{z_2}{M} : D(0, M) \to \mathbb{D}$$

$$\varphi_3(z_3) = \frac{f(0)/M - z_3}{1 - \overline{f(0)}/M z_3} : \mathbb{D} \to \mathbb{D},\ f(0)/M \mapsto 0$$

Thus the function

$$\varphi := \varphi_3 \circ \varphi_2 \circ f \circ \varphi_1 = \frac{\frac{f(0)}{M} - \frac{f(Rz_1)}{M}}{1 - \frac{\overline{f(0)}}{M} \cdot \frac{f(Rz_1)}{M}}$$

defines a function sending $\mathbb{D}$ to $\mathbb{D}$ and sending 0 to 0. Therefore, by Schwarz's lemma

$$\left|\frac{\frac{f(0)}{M} - \frac{f(Rz_1)}{M}}{1 - \frac{\overline{f(0)}}{M} \cdot \frac{f(Rz_1)}{M}}\right| \leq |z_1|$$

after change variable $z = Rz_1$, we have

$$\left|\frac{f(0) - f(z)}{1 - \overline{f(0)} \cdot f(z)}\right| \leq \frac{|z|}{RM}$$

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3. (Ex.14 Ch.8 in textbook) Prove that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ take the form:

$$e^{i\theta} \frac{z - \beta}{z - \overline{\beta}}$$

Solution. Let φ be such a conformal mapping, and we have a conformal mapping from $\mathbb{H}$ to $\mathbb{D}$:

$$\psi(z) = \frac{z - i}{z + i}$$

then $\varphi \circ \psi^{-1}$ is a conformal mapping from $\mathbb{D}$ to $\mathbb{D}$ (i.e an automorphism of $\mathbb{D}$). Thus $\varphi \circ \psi^{-1}$ must take the form

$$e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z} \quad \text{for some} \alpha$$

This tells us φ must be

$$e^{i\theta} \frac{\alpha - \frac{z-i}{z+i}}{1 - \overline{\alpha}\frac{z-i}{z+i}} = e^{i\theta} \frac{\alpha(z+i) - z + i}{z + i - \overline{\alpha}(z - i)} = e^{i\theta} \frac{z(\alpha - 1) + i(\alpha + 1)}{z(1 - \overline{\alpha}) + i(1 + \overline{\alpha})} = e^{i\theta'} \frac{z + i\frac{\alpha+1}{\alpha-1}}{z + i\frac{1+\overline{\alpha}}{1-\overline{\alpha}}}$$

If we let $\beta = i\frac{\alpha+1}{\alpha-1}$, we get the result we desired. ◀